INTEGRABLE SPHERICALLY SYMMETRIC P-BRANE MODELS ASSOCIATED WITH LIE ALGEBRAS

S. Cotsakis[1]**†V. R. Gavrilov**[2]**‡, V. N. Melnikov**[3]**‡**

† Research Laboratory for Geometry, Dynamical Systems and Cosmology (GEODYSYC),
Department of Mathematics University of the Aegean, Karlovassi, 83 200 Samos, Greece

‡ Centre for Gravitation and Fundamental Metrology
VNIIMS, 3-1 M. Ulyanovoy St., Moscow 117313, Russia

Abstract

A classical model of gravity theory with several dilatonic scalar fields and differential forms admitting an interpretation in terms of intersecting p-branes is studied in (pseudo)-Riemannian space-time manifold $\mathrm{M} = \mathbb{R}_+ \times S^{d_0} \times \mathbb{R}_t \times M_2^{d_2} \ldots \times M_n^{d_n}$, $(n \geq 2)$ of dimension D. The equations of motion of the model are reduced to the Euler-Lagrange equations for the so-called pseudo-Euclidean Toda-like system. We suppose that the characteristic vectors related to the configuration of p-branes and their couplings to the dilatonic scalar fields may be interpreted as the root vectors of a Lie algebra of the types A_r, B_r, C_r. In this case the model is reducible to one of the open Toda chain's algebraic generalization and is completely integrable by the known methods. The corresponding general solutions are presented in explicit form. The particular exact solution describing a class of nonextremal black holes is obtained and analyzed.

PACS: 04.50.+h, 98.80.Hw, 04.60.Kz Keywords: p-branes, spherical symmetry, black holes

1 Introduction

There has been much interest in black hole solutions in p-brane theory [1]-[25] because of the possible resolution of various puzzles associated with quantum gravity[14]. A growth of interest in classical p-brane solutions of supergravities of various dimensions is inspired by a conjecture that $D = 11$ supergravity is a low-energy effective field theory of eleven-dimensional fundamental M-theory, which (together with so called F-theory) is a candidate for unification of five known ten-dimensional superstring models. Classical p-brane solutions may be considered as an instrument for investigation of interlinks between superstrings and M-theory.

In this paper we consider generalized bosonic sector (without Chern-Simons terms) of supergravity theories [26] in the form of a multidimensional gravitational model with several dilatonic scalar fields and differential forms of various ranks admitting an interpretation in terms of intersecting p-branes. As was shown in [15], for cosmological and static spherically symmetric space-times the equations of motion of such a model are reduced to the Euler-Lagrange equations for the so-called pseudo-Euclidean Toda-like Lagrange system. We reproduce this result in Sec. 2. Methods for integrating of pseudo-Euclidean Toda-like systems (see, [27] and references therein) are based on a Minkowski-like geometry for the characteristic vectors determining the potential of the pseudo-Euclidean Toda-like system. If the characteristic vectors form an orthogonal set, then the pseudo-Euclidean Toda-like system is integrable. The corresponding

---

[1]e-mail: skot@aegean.gr
[2]e-mail: gavr@rgs.phys.msu.su
[3]e-mail: melnikov@rgs.phys.msu.su

p-brane models have been studied in [15],[19],[20],[23]. Here we apply these methods for integrating the p-brane model reducible to the algebraic generalizations of an open Toda chain. The characteristic vectors of such models may be interpreted as root vectors of the semi-simple Lie algebra. We consider the Lie algebras of the Cartan types A_r, B_r, C_r. Using the technique suggested by Anderson [29] for solving the Toda chain's equations of motion, in Sec. 3,4 we integrate the p-brane models. In the last section of the paper we examine the metric obtained for some particular exact solution appearing for a quite wide class of p-brane models. This solution describes the nonextremal black hole under some condition. The corresponding ADM-mass and the Hawking temperature of a black hole are calculated.

2   The general model

Following the papers [15],[17],[23] we consider here a classical model of gravity theory with several dilatonic scalar fields φ^α and differential n_a-forms $F^a_{M_1...M_{n_a}}$ in (pseudo)-Riemannian space-time manifold M of dimension D. The action of the model reads

$$S = \int_M d^D z \sqrt{|g|} \left(R[g] - \sum_{\alpha,\beta=1}^{\omega} h_{\alpha\beta}\, g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a\in\Delta} \frac{e^{2\lambda_a(\varphi)}}{n_a!} (F^a)^2 \right), \qquad (2.1)$$

where $ds^2 = g_{MN} dz^M dz^N$ is the metric with Lorentzian signature on the manifold M ($M, N = 0, 1 \ldots, D-1$), $|g| = |\det(g_{MN})|$. $(h_{\alpha\beta})$ is a symmetrical positively definite $\omega \times \omega$ matrix, $\lambda_a(\varphi)$ is a linear combination of the scalar fields, i.e.

$$\lambda_a(\varphi) = \sum_{\alpha=1}^{\omega} \lambda_{a,\alpha} \varphi^\alpha, \qquad (2.2)$$

where $\lambda_{a,\alpha}$ are the coupling constants. Furthermore

$$F^a = \frac{1}{n_a!} F^a_{M_1...M_{n_a}} dz^{M_1} \wedge \ldots \wedge dz^{M_{n_a}} = dA^a, \qquad (2.3)$$

$$A^a = \frac{1}{(n_a-1)!} A^a_{M_1...M_{n_a-1}} dz^{M_1} \wedge \ldots \wedge dz^{M_{n_a-1}}, \qquad (2.4)$$

$$(F^a)^2 = F^a_{M_1...M_{n_a}} F^a_{N_1...N_{n_a}} g^{M_1 N_1} \ldots g^{M_{n_a} N_{n_a}}. \qquad (2.5)$$

The field $A^a_{M_1,...,M_{n_a-1}}$ may be called a gauge potential corresponding to the field strength $F^a_{M_1,...,M_{n_a}}$. By Δ we denote some finite set.
    The action (2.1) leads to the following equations of motion

$$R_{MN} - \frac{1}{2} g_{MN} R = T_{MN}, \qquad (2.6)$$

$$\triangle[g]\varphi^\alpha = \sum_{a\in\Delta} \frac{1}{n_a!} \lambda_a^\alpha e^{2\lambda_a(\varphi)} (F^a)^2, \quad \alpha = 1, \ldots, \omega, \qquad (2.7)$$

$$\nabla_{M_1}[g](e^{2\lambda_a(\varphi)} F^{a,M_1...M_{n_a}}) = 0, \quad a \in \Delta. \qquad (2.8)$$

The right side of the Einstein equations (2.6) looks as follows

$$T_{MN} = T_{MN}[\varphi] + T_{MN}[F], \qquad (2.9)$$

where we denoted

$$T_{MN}[\varphi] = \sum_{\alpha,\beta=1}^{\omega} h_{\alpha\beta} \left(\partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{2} g_{MN} \partial_P \varphi^\alpha \partial^P \varphi^\beta \right), \tag{2.10}$$

$$T_{MN}[F] = \sum_{a\in\Delta} \frac{e^{2\lambda_a(\varphi)}}{n_a!} \left(-\frac{1}{2} g_{MN}(F^a)^2 + n_a F^a_{MM_2...M_{n_a}} F^{a,M_2...M_{n_a}}_N \right). \tag{2.11}$$

In (2.7),(2.8) we denoted the Laplace-Beltrami operator and covariant derivative with respect to the metric g_{MN} by $\triangle[g]$ and $\bigtriangledown_M[g]$, respectively. The constants λ_a^α in (2.7) are introduced by

$$\lambda_a^\alpha = \sum_{\alpha,\beta=1}^{\omega} h^{\alpha\beta} \lambda_{a,\beta}, \tag{2.12}$$

where $(h^{\alpha\beta})$ is the inverse matrix to $(h_{\alpha\beta})$.

Consider the model introduced under the following assumptions. Let the D-dimensional space-time M be decomposed into the direct product of $\mathbb{R}_+$ (corresponding to a radial coordinate u), d_0-dimensional sphere S^{d_0} ($d_0 \geq 2$), time axis $\mathbb{R}_t$ and $(n-1)$ factor spaces $M_2, \ldots, M_n$, i.e.

$$M = \mathbb{R}_+ \times S^{d_0} \times \mathbb{R}_t \times M_2^{d_2} \ldots \times M_n^{d_n}, \quad n \geq 2. \tag{2.13}$$

The metric on M is assumed correspondingly to be

$$ds^2 = e^{2\gamma(u)} du^2 + e^{2x^0(u)} d\Omega_{d_0}^2 - e^{2x^1(u)} dt^2 + \sum_{i=2}^{n} e^{2x^i(u)} ds_i^2, \tag{2.14}$$

where u is the radial coordinate, $d\Omega_{d_0}^2 = g^0_{m_0 n_0}(y_0) dy_0^{m_0} dy_0^{n_0}$ is the line element on d_0-dimensional unit sphere, t is the time coordinate, $ds_i^2 = g^i_{m_i n_i}(y_i) dy_i^{m_i} dy_i^{n_i}$ is the positively definite metric on the d_i-dimensional factor space M_i, $\gamma(u), x^2(u), \ldots, x^n(u)$ are scalar functions of the radial coordinate u. Herein, for reasons of simplicity, only Ricci-flat spaces $M_2, \ldots, M_n$ are assumed (i.e. the components of the Ricci tensor for the metrics $g^i_{m_i n_i}$ are zero).

It is useful to consider S^{d_0} and $\mathbb{R}_t$ as factor spaces M_0 and M_1, respectively. So we put

$$M_0 \equiv S^{d_0}, \tag{2.15}$$

$$M_1 \equiv \mathbb{R}_t, \ d_1 = 1. \tag{2.16}$$

We split the coordinates on M into the following ranges:

$$\left(z^0, z^1, \ldots, z^{d_0}, z^{d_0+1}, \ldots, z^{D-d_n}, \ldots, z^{D-1} \right) = \left(u, y_0^1, \ldots, y_0^{d_0}, t, \ldots, y_n^1, \ldots, y_n^{d_n} \right). \tag{2.17}$$

We introduce the following d_i-forms on M

$$\tau_1 = dt, \ \tau_i = \sqrt{\det(g^i_{m_i n_i})} dy_i^1 \wedge \ldots \wedge dy_i^{d_i}, \quad i = 0, 2, \ldots, n. \tag{2.18}$$

Clearly, the canonical projection $\hat{p}_i : M \to M_i$ of τ_i provides with the volume form of M_i.

In order to construct the p-brane worldvolumes we introduce submanifolds of the following type

$$M_I = M_{i_1} \times \ldots \times M_{i_r}, \tag{2.19}$$

where

$$I = \{i_1, \ldots, i_r\}, \quad i_1 < \ldots < i_r, \tag{2.20}$$

3

is any ordered non-empty subset of natural numbers $2, \ldots, n$. Let Ω_0 be the set of all such elements including the empty set, i.e.

$$\Omega_0 = \{\emptyset, \{2\}, \{3\}, \ldots, \{n\}, \{2, 3\}, \ldots, \{2, 3, \ldots, n\}\}. \tag{2.21}$$

By definition, put

$$\bar{I} \equiv \{2, \ldots, n\} \setminus I. \tag{2.22}$$

In this paper we consider electrically charged p-branes with the following worldvolumes

$$M_I^{(e)} = \mathbb{R}_t \times M_I, \ I = \{i_1, \ldots, i_r\} \in \Omega_0. \tag{2.23}$$

For empty $I = \emptyset$ we put $M_I^{(e)} = \mathbb{R}_t$. The dimension of $M_I^{(e)}$ is given by

$$d(I) \equiv \dim M_I^{(e)} = 1 + d_{i_1} + \ldots + d_{i_r}. \tag{2.24}$$

$(d(I) = 1$ for $I = \emptyset)$. The canonical projection $\hat{p}_I : M \to M_I^{(e)}$ of the following $d(I)$-form

$$\tau(I) = dt \wedge \tau_{i_1} \wedge \ldots \wedge \tau_{i_r} \tag{2.25}$$

is the volume form of $M_I^{(e)}$. We put $\tau(I) = dt$ for $I = \emptyset$.

In accordance with the terminology of p-brane theory [1] an $(n_a - 1)$-form potential

$$A^{(a,e,I)} = \Phi^{(a,e,I)}(u)\tau(I), \ \mathrm{rank}A^{(a,e,I)} \equiv n_a - 1 = d(I), \ a \in \Delta, \tag{2.26}$$

where $\Phi^{(a,e,I)}(u)$ is a scalar function, describes an electrically charged p-brane $(p = n_a - 2)$ with the worldvolume $M_I^{(e)}$. Moreover, the submanifold $\mathbb{R}_+ \times S^{d_0} \times M_{\bar{I}}$ $(\mathbb{R}_+ \times S^{d_0}$ for $I = \{2, \ldots, n\})$ is the so-called transverse space for this p-brane. The n_a-form field strength corresponding to $A^{(a,e,I)}$ was defined by (2.3) and may be written as

$$F^{(a,e,I)} = d\Phi^{(a,e,I)}(u) \wedge \tau(I) = \dot{\Phi}^{(a,e,I)}(u)du \wedge \tau(I). \tag{2.27}$$

The overdot means a derivative with respect to the radial coordinate u.

An n_b-form field strength

$$F^{(b,m,J)} = \mathrm{e}^{-2\lambda_b(\varphi)} * \left(d\Phi^{(b,m,J)}(u) \wedge \tau(J)\right), \ J \in \Omega_0, \ b \in \Delta, \tag{2.28}$$

describes a p-brane $(p = n_b - 1 = D - d(J) - 2)$ with a magnetic-type charge. The submanifold

$$M_J^{(m)} = S^{d_0} \times M_{\bar{J}} \tag{2.29}$$

is a worldvolume of this p-brane. Clearly, $M_J^{(m)} = S^{d_0}$ for $J = \{2, \ldots, n\}$. By $*$ we denoted the Hodge operator on the manifold (M, g), i.e.

$$(*F)_{M_1 \ldots M_{D-r}} = \frac{\sqrt{|g|}}{r!} \varepsilon_{N_1 \ldots N_r M_1 \ldots M_{D-r}} F^{N_1 \ldots N_r}. \tag{2.30}$$

In this paper we consider the so-called composite p-branes [9], i.e., by definition we put

$$F^a = \sum_{I \in \Omega_{a,e}} F^{(a,e,I)} + \sum_{J \in \Omega_{a,m}} F^{(a,m,J)}, \tag{2.31}$$

4

where $\Omega_{a,e} \subset \Omega_0$ is a subset (which may be empty) of all $I \in \Omega_0$ such that $d(I) + 1 = n_a \equiv \mathrm{rank} F^{(a,e,I)}$. Moreover, $\Omega_{a,m} \subset \Omega_0$ is a subset (which may be empty) of all $J \in \Omega_0$ such that $D - d(J) - 1 = \dim M_J^{(m)} = n_a \equiv \mathrm{rank} F^{(a,m,J)}$. Evidently, $\Omega_{a,m} = \emptyset$ for $n_a = D - 1, D$.

We obtain the following non-zero components of the Ricci tensor for the metric (2.14)

$$R_0^0 = -\mathrm{e}^{-2\gamma} \left(\sum_{k=0}^{n} d_k (\dot{x}^k)^2 + \ddot{\gamma}_0 - \dot{\gamma}\dot{\gamma}_0 \right), \tag{2.32}$$

$$R_{n_k}^{m_k} = \left\{ \delta_0^k (d_0 - 1) \mathrm{e}^{-2x^k} - \left[\ddot{x}^k + \dot{x}^k (\dot{\gamma}_0 - \dot{\gamma}) \right] \mathrm{e}^{-2\gamma} \right\} \delta_{n_k}^{m_k}, \tag{2.33}$$

where we denoted

$$\gamma_0 = \sum_{k=0}^{n} d_k x^k. \tag{2.34}$$

Indices m_k and n_k in (2.33) for $k = 0, \ldots, n$ run over from $(D - \sum_{l=k}^{n} d_l)$ to $(D - \sum_{l=k}^{n} d_l + d_k - 1)$. We recall that $D = 1 + \sum_{k=0}^{n} d_k = \dim M$.

Under the above assumptions related to the F^a-fields and the metric (2.14) the Maxwell-like equations (2.8) and the Bianchi identities $dF^a = 0$ have the following form, correspondingly

$$\frac{d}{du} \left[\mathrm{e}^{\gamma_0 - \gamma - 2\sigma(I) + 2\lambda_a(\phi)} \dot{\Phi}^{(a,e,I)}(u) \right] = 0, \ I \in \Omega_{a,e}, \ a \in \Delta \tag{2.35}$$

$$\frac{d}{du} \left[\mathrm{e}^{\gamma_0 - \gamma - 2\sigma(J) - 2\lambda_a(\phi)} \dot{\Phi}^{(a,m,J)}(u) \right] = 0, \ J \in \Omega_{a,m}, \ a \in \Delta, \tag{2.36}$$

where

$$\sigma(I) = d_1 x^1 + \sum_{i \in I} d_i x^i. \tag{2.37}$$

For empty $I = \emptyset$ we put $\sigma(I) = d_1 x^1$.

To denote F^a-fields and their potentials, it is useful the following collective index

$$s = (a, v, I), \ I \in \Omega_{a,v}, \ v = e, m, \ a \in \Delta. \tag{2.38}$$

By S we denote the set of all elements s, i.e.

$$S = \bigsqcup_{v=e,m} \left(\bigsqcup_{a \in \Delta} \{a\} \times \{v\} \times \Omega_{a,v} \right). \tag{2.39}$$

Integrating (2.35) and (2.36), we get

$$\dot{\Phi}^s(u) = Q_s \exp \left[\gamma - \gamma_0 + 2\sigma(I_s) - 2\chi_s \lambda_{a_s}(\phi) \right], \ s = (a_s, v_s, I_s) \in S, \tag{2.40}$$

where

$$\chi_s = +1, \ v_s = e, \tag{2.41}$$

$$\chi_s = -1, \ v_s = m. \tag{2.42}$$

Q_s are arbitrary constants.

Let $S_* \subset S$ be a subset of all $s \in S$ such that $Q_s \neq 0$.

To obtain the tensors $T[F^a]_N^M$ in a block-diagonal form, we put the following restriction: there are no elements $(a, v, I), (a, v, J) \in S_*$ such that

$$I = (I \cap J) \sqcup \{i\}, \ J = (I \cap J) \sqcup \{j\}, \ i \neq j, \ d_i = d_j = 1, \tag{2.43}$$

where $i, j = 2, \ldots, n$. Here the intersection $I \cap J$ may be empty. The total energy-momentum tensor of F^a-fields has a block-diagonal form whenever the restriction (2.43) is valid. Using (2.40), we present its non-zero components in the form

$$T[F^a]^0_0 = -\frac{1}{2} e^{-2\gamma_0} \sum_{s \in S_*} Q_s^2 \exp\left[2\sigma(I_s) - 2\chi_s \lambda_{a_s}(\phi)\right], \qquad (2.44)$$

$$T[F^a]^{m_k}_{n_k} = -\frac{1}{2} e^{-2\gamma_0} \left(\sum_{s \in S_*} (2\delta_{kI_s} - 1) Q_s^2 \exp\left[2\sigma(I_s) - 2\chi_s \lambda_{a_s}(\phi)\right] \right) \delta^{m_k}_{n_k}, \qquad (2.45)$$

where

$$\delta_{kI} = \sum_{i \in I} \delta_{ki} + \delta_{k1}, \ I \in \Omega_0, \ k = 0, 1, \ldots, n. \qquad (2.46)$$

We put $\delta_{kI} = \delta_{k1}$ for $I = \emptyset$. Evidently, $\delta_{0I} = 0$ and $\delta_{1I} = 1$ for any $I \in \Omega_0$.

We assume that the dilatonic scalar fields φ^α depend only on the radial coordinate u. Under this assumption the total energy-momentum tensor of the dilatonic scalar fields reads

$$\left(T[\varphi]^M_N \right) = \frac{1}{2} e^{-2\gamma} \left(\sum_{\alpha,\beta=1}^{\omega} h_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta \right) \text{diag}(1, \underbrace{1, \ldots, 1}_{d_0 \text{ times}}, -1, 1, \ldots, 1). \qquad (2.47)$$

The Einstein equations (2.6) can be written as $R^M_N = T^M_N - T\delta^M_N/(D-2)$. Further we employ the equations $R^0_0 - R/2 = T^0_0$ and $R^{m_k}_{n_k} = T^{m_k}_{n_k} - T\delta^{m_k}_{n_k}/(D-2)$. Using (2.32),(2.33),(2.44)(2.45),(2.47), we obtain these equations in the form

$$\frac{1}{2} \left(\sum_{k,l=0}^{n} G_{kl} \dot{x}^k \dot{x}^l + \sum_{\alpha,\beta=1}^{\omega} h_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta \right) + V = 0, \qquad (2.48)$$

$$\ddot{x}^k + (\dot{\gamma}_0 - \dot{\gamma})\dot{x}^k = e^{2\gamma} \left\{ \delta^k_0 (d_0 - 1) e^{-2x^k} \right.$$
$$\left. + e^{-2\gamma_0} \sum_{s \in S_*} \chi_s \left(\delta_{kI_s} - \frac{d(I_s)}{D-2} \right) Q_s^2 \exp\left[2\sigma(I_s) - 2\chi_s \lambda_{a_s}(\phi)\right] \right\} \qquad (2.49)$$

where we denoted

$$G_{kl} = d_k \delta_{kl} - d_k d_l, \ k, l = 0, 1, \ldots, n, \qquad (2.50)$$

$$V = \frac{1}{2} e^{2\gamma} \left\{ (d_0 - 1) d_0 e^{-2x^0} - e^{-2\gamma_0} \sum_{s \in S_*} Q_s^2 \exp\left[2\sigma(I_s) - 2\chi_s \lambda_{a_s}(\phi)\right] \right\}. \qquad (2.51)$$

Under the above assumptions equations (2.7) have the form

$$\ddot{\varphi}^\alpha + (\dot{\gamma}_0 - \dot{\gamma})\dot{\varphi}^\alpha = e^{2\gamma - 2\gamma_0} \sum_{s \in S_*} \chi_s \lambda^\alpha_{a_s} Q_s^2 \exp\left[2\sigma(I_s) - 2\chi_s \lambda_{a_s}(\phi)\right]. \qquad (2.52)$$

It is not difficult to verify that equations (2.48),(2.49),(2.53) may be presented as the Euler-Lagrange equations obtained from the Lagrangian

$$L = e^{\gamma_0 - \gamma} \left[\frac{1}{2} \left(\sum_{k,l=0}^{n} G_{kl} \dot{x}^k \dot{x}^l + \sum_{\alpha,\beta=1}^{\omega} h_{\alpha\beta} \dot{\varphi}^\alpha \dot{\varphi}^\beta \right) - V \right] \qquad (2.53)$$

6

viewed as a function of the generalized coordinates $\gamma, x^k, \varphi^\alpha$. The equation $\partial L/\partial\gamma = d(\partial L/\partial\dot\gamma)/du$ leads to the zero-energy constraint (2.48). After the gauge fixing: $\gamma = F(x^k, \varphi^\alpha)$ the equations (2.49),(2.52) may be considered as the Euler-Lagrange equations obtained from the Lagrangian (2.53) under the constraint (2.48). Further, we use the so-called harmonic gauge

$$\gamma \equiv \gamma_0 = \sum_{k=0}^{n} d_k x^k. \tag{2.54}$$

It is easy to check that the radial coordinate u is a harmonic function in this gauge, i.e. $\triangle[g]u = 0$.

Let us introduce an $(n+\omega+1)$-dimensional real vector space $\mathbb{R}^{n+\omega+1}$. Denote by e_A, $A = 0, 1, \ldots, n+\omega$, the canonical basis in $\mathbb{R}^{n+\omega+1}$ ($e_1 = (1,0,\ldots,0)$ etc.). Define the following vectors:

1. The vector whose coordinates are to be found

$$x(u) = \sum_{A=0}^{n+\omega} x^A(u)e_A, \quad \left(x^A(u)\right) = (x^0(u), \ldots, x^n(u), \varphi^1(u), \ldots, \varphi^\omega(u)). \tag{2.55}$$

2. The vector corresponding to the factor-space $M_0 \equiv S^{d_0}$ with a non-zero Ricci tensor. Hereafter, we call it by the vector induced by the curvature of M_0

$$V_0 = \sum_{A=0}^{n+\omega} V_0^A e_A = -\frac{2}{d_0}e_0, \quad \left(V_0^A\right) = -2\left(\frac{1}{d_0}, 0, \ldots, 0\right). \tag{2.56}$$

3. The vector induced by a p-brane

$$U_s = \sum_{A=0}^{n+\omega} U_s^A e_A, \quad \left(U_s^A\right) = 2\left(\underbrace{\delta_{kI} - d(I_s)/(D-2)}_{n+1}, \underbrace{-\chi_s\lambda_{a_s}^\alpha}_{\omega}\right). \tag{2.57}$$

A set of the vectors U_s characterizes the space-time M, the configuration of p-branes and their couplings to dilatonic scalar fields. Further these vectors are called characteristic.

Let $< .,. >$ be a symmetrical bilinear form on $\mathbb{R}^{n+\omega+1}$ such that

$$< e_A, e_B > = \bar{G}_{AB}, \tag{2.58}$$

where we put by definition

$$(\bar{G}_{AB}) = \begin{pmatrix} G_{kl} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix}. \tag{2.59}$$

The form is nondegenerate, the inverse matrix to (G_{AB}) reads

$$(\bar{G}^{AB}) = \begin{pmatrix} \frac{\delta^{kl}}{d_k} + \frac{1}{2-D} & 0 \\ 0 & h^{\alpha\beta} \end{pmatrix}. \tag{2.60}$$

The form $< .,. >$ endows the space $\mathbb{R}^{n+\omega+1}$ with the metric, whose signature is $(-, +, \ldots, +)$ [15]. By the usual way we may introduce the covariant components of vectors. For the vectors

V_0, U_s the covariant components have the form

$$V_{0,A} = 2(\underbrace{d_k - \delta_{0k}}_{n+1}, \underbrace{0, \ldots, 0}_{\omega}), \tag{2.61}$$

$$U_{s,A} = 2(\underbrace{d_k \delta_{kI}}_{n+1}, \underbrace{-\chi_s \lambda_{a_s, \alpha}}_{\omega}), \tag{2.62}$$

The values of the bilinear form $< .,. >$ for V_0, U_s look as follows

$$< V_0, V_0 >= -4\frac{d_0 - 1}{d_0}, \tag{2.63}$$

$$< V_0, U_s >= 0, \ \forall s \in S, \tag{2.64}$$

$$< U_s, U_{s'} >= 4 \left[d(I_s \cap I_{s'}) - \frac{d(I_s)d(I_{s'})}{D - 2} + \chi_s \chi_{s'} \sum_{\alpha,\beta=1}^{\omega} h_{\alpha\beta} \lambda_{a_s}^{\alpha} \lambda_{a_{s'}}^{\beta} \right], \tag{2.65}$$

where $s = (a_s, v_s, I_s), s' = (a_{s'}, v_{s'}, I_{s'}) \in S$.

A vector $y \in R^{n+\omega+1}$ is called time-like, space-like or isotropic, if $< y, y >$ has negative, positive or null values respectively. Vectors y and z are called orthogonal if $< y, z >= 0$. It should be noted that the curvature induced vector V_0 is always time-like, while the p-brane induced vector U_s admits any value of $< U_s, U_s >$. We mention that V_0 and U_s are always orthogonal.

Using the notation $< .,. >$ and the vectors (2.55)-(2.57), we represent the Lagrangian (2.53) and the zero-energy constraint (2.48) with respect to a harmonic time gauge in the form

$$L = \frac{1}{2} < \dot{x}, \dot{x} > -V, \tag{2.66}$$

$$E = \frac{1}{2} < \dot{x}, \dot{x} > +V \equiv 0, \tag{2.67}$$

where the potential V reads

$$V = a^{(0)} e^{<V_0, x>} + \sum_{s \in S_*} a^{(s)} e^{<U_s, x>}. \tag{2.68}$$

The following notation is used

$$a^{(0)} = \frac{d_0(d_0 - 1)}{2}, \ a^{(s)} = -\frac{1}{2}Q_s^2, \ s \in S_*. \tag{2.69}$$

¿From the mathematical viewpoint the obtaining of exact solutions in the p-brane model under consideration is reduced to integration of equations of motion for a system with $(n+\omega+1)$ degrees of freedom described by the Lagrangian of the form

$$L = \frac{1}{2} < \dot{x}, \dot{x} > - \sum_{\mu=1}^{r} a^{(\mu)} e^{<b_\mu, x>}, \tag{2.70}$$

where $x, b_\mu \in \mathbb{R}^{n+\omega+1}$. It should be noted that the kinetic term $< \dot{x}, \dot{x} >$ is not a positively definite quadratic form as there usually takes place in classical mechanics. Due to the pseudo-Euclidean signature $(-, +, ..., +)$ of the form $< .,. >$ such systems may be called the pseudo-Euclidean Toda-like systems as the potential like that given in (2.70) defines the algebraic generalizations of the Toda chain [28]. well-known in classical mechanics .

8

3 Integration of the p-brane model with linearly independent characteristic vectors

We recall that $S_* \subset S$ is the subset of all $s \in S$ such that $Q_s \neq 0$. Define a bijection $f : S_* \mapsto \{1, 2, \ldots, r\}$, where we denote by r the cardinal number of S_*, i.e.

$$r = |S_*|. \tag{3.1}$$

Denote the natural number $f(s)$ corresponding to $s \in S_*$ by the same letter s.

The problem consists in integrating the equations of motion obtained from the Lagrangian (2.66) under the zero-energy constraint (2.67). Suppose the characteristic vectors $U_s \in \mathbb{R}^{n+\omega+1}$, induced by p-branes are linearly independent. Then $r \leq n + \omega$. We introduce a basis $\{f_A\}$ in $\mathbb{R}^{n+\omega+1}$ in the following manner

$$f_0 = \frac{V_0}{< V_0, V_0 >}, \;\; f_s = \frac{2U_s}{< U_s, U_s >}, \;\; s = 1, \ldots, r, \tag{3.2}$$

$$< f_A, f_p > = \delta_{Ap}, \;\; A = 0, \ldots, n + \omega, p = r + 1, \ldots, n + \omega. \tag{3.3}$$

Notice that if $r \equiv |S_*| = n + \omega$ then the basis $\{f_A\}$ does not contain the vectors f_p with $p \geq r+1$. We also mention that due to the relation (2.64) we get $< f_0, f_s > = 0$ for $s = 1, \ldots, r$. It is not difficult to prove that the vectors $f_1, \ldots, f_r, f_{r+1}, \ldots, f_{n+\omega}$ must be space-like.

Using the decomposition

$$x(u) = q^0(u)f_0 + \sum_{s=1}^{r}[q^s(u) - \ln C^s]f_s + \sum_{p=r+1}^{n+\omega} q^p(u)f_p, \tag{3.4}$$

we present the Lagrangian (2.66) and the constraint (2.67) in the form

$$L = L_0 + L_T + L_P, \tag{3.5}$$

$$E = E_0 + E_T + E_P \equiv 0, \tag{3.6}$$

where

$$L_0 = \frac{(\dot{q}^0)^2}{2 < V_0, V_0 >} - \frac{d_0(d_0 - 1)}{2} e^{q^0}, \tag{3.7}$$

$$E_0 = \frac{(\dot{q}^0)^2}{2 < V_0, V_0 >} + \frac{d_0(d_0 - 1)}{2} e^{q^0}, \tag{3.8}$$

$$L_T = \sum_{s,s'=1}^{r} \frac{C_{ss'}}{< U_s, U_s >} \dot{q}^s \dot{q}^{s'} + \sum_{s=1}^{r} \frac{2}{< U_s, U_s >} \exp\left[\sum_{s'=1}^{r} C_{ss'} q^{s'}\right], \tag{3.9}$$

$$E_T = \sum_{s,s'=1}^{r} \frac{C_{ss'}}{< U_s, U_s >} \dot{q}^s \dot{q}^{s'} - \sum_{s=1}^{r} \frac{2}{< U_s, U_s >} \exp\left[\sum_{s'=1}^{r} C_{ss'} q^{s'}\right], \tag{3.10}$$

$$L_P = E_P = \frac{1}{2} \sum_{p=r+1}^{n+\omega} (\dot{q}^p)^2. \tag{3.11}$$

We introduced, in (3.9),(3.10), the nondegenerate Cartan-type matrix $(C_{ss'})$ by the following manner

$$C_{ss'} = 2\frac{< U_s, U_{s'} >}{< U_{s'}, U_{s'} >}, \;\; s, s' = 1, \ldots, r. \tag{3.12}$$

The constants C^s in the decomposition (3.4) are defined by

$$C^s = \prod_{s'=1}^{r} \left[\frac{< U_{s'}, U_{s'} >}{4} Q_{s'}^2 \right]^{C^{ss'}}, \quad s = 1, \ldots, r, \tag{3.13}$$

where $(C^{ss'})$ is the inverse matrix to $(C_{ss'})$.

The Euler-Lagrange equations for $q^{r+1}(u), \ldots, q^{n+\omega}(u)$ read $\ddot{q}^{r+1}(u) = \ldots = \ddot{q}^{n+\omega}(u) = 0$. Integrating them, we get

$$q^p(u) = a^p u + b^p, \; p = r + 1, \ldots, n + \omega, \tag{3.14}$$

$$E_P = \frac{1}{2} \sum_{p=r+1}^{n+\omega} (a^p)^2 \geq 0, \tag{3.15}$$

where the constants a^p, b^p are arbitrary.

For $q^0(u)$ we get the Liouville equation. The result of its integration reads

$$e^{-q^0(u)/2} = F_0(u - u_0), \tag{3.16}$$

where u_0 is an arbitrary constant. The function F_0 is defined by

$$F_0(u) = \frac{\sin\left[\sqrt{\frac{2E_0}{d_0(d_0-1)}}(d_0 - 1)u\right]}{\sqrt{\frac{2E_0}{d_0(d_0-1)}}}. \tag{3.17}$$

This representation implies

$$F_0(u) \quad = (d_0 - 1)u, \; E_0 = 0, \tag{3.18}$$

$$= \frac{\sin\left[\sqrt{\frac{2E_0}{d_0(d_0-1)}}(d_0 - 1)u\right]}{\sqrt{\frac{2E_0}{d_0(d_0-1)}}}, \; E_0 > 0 \tag{3.19}$$

$$= \frac{\sinh\left[\sqrt{\frac{2|E_0|}{d_0(d_0-1)}}(d_0 - 1)u\right]}{\sqrt{\frac{2|E_0|}{d_0(d_0-1)}}}, \; E_0 < 0. \tag{3.20}$$

The equations of motion for $q^1(u), \ldots, q^r(u)$ look as follows

$$\ddot{q}^s = \exp\left[\sum_{s'=1}^{r} C_{ss'} q^{s'}\right], \; s, = 1, \ldots, r. \tag{3.21}$$

Using the transformation

$$F_s(u) = e^{-q^s(u)}, \tag{3.22}$$

we present the set of equations (3.21) in the form

$$\dot{F}_s^2 - F_s \ddot{F}_s = F_s^2 \prod_{s'=1}^{r} (F_{s'})^{-C_{ss'}}. \tag{3.23}$$

The set of equations (3.21) proved to be completely integrable if $(C_{ss'})$ is the Cartan matrix of a simple complex Lie algebra. The general solutions for some algebras as well as some particular

10

solution of the set (3.21) for quite a wide class of matrices $(C_{ss'})$ will be considered in the next sections. Here we suppose that the functions are known and the corresponding integral of motion (3.10) is calculated.

Combining (3.2),(3.14),(3.16)(3.22), we present the decomposition (3.4) in the following form

$$x(u) = -\ln[F_0(u-u_0)]\frac{2V_0}{<V_0,V_0>} - \sum_{s=1}^{r}\ln[C^s F_s(u)]\frac{2U_s}{<U_s,U_s>} + uQ + P, \tag{3.24}$$

where vectors $Q, P \in \mathbb{R}^{n+\omega+1}$ are defined by

$$Q = \sum_{p=r+1}^{n+\omega} a^p f_p \equiv \sum_{A=0}^{n+\omega} Q^A e_A, \; P = \sum_{p=r+1}^{n+\omega} b^p f_p \equiv \sum_{A=0}^{n+\omega} P^A e_A, \tag{3.25}$$

Due to the assumptions (3.3) their coordinates Q^A, P^A w.r.t. the canonical basis $\{e_A\}$ satisfy the constraints

$$<Q,V_0> = 2\sum_{k=0}^{n} Q^k(d_k - \delta_{k0}) = 0, \; <P,V_0> = 2\sum_{k=0}^{n} P^k(d_k - \delta_{k0}) = 0. \tag{3.26}$$

$$<Q,U_s> = \sum_{A=0}^{n+\omega} Q^A U_{s,A} = 0, \; <P,U_s> = \sum_{A=0}^{n+\omega} P^A U_{s,A} = 0, \; s = 1,\dots,r. \tag{3.27}$$

Finally, the exact solution can be summarized as follows.

1. The metric (2.14) in the harmonic time gauge (2.54) reads

$$ds^2 = \prod_{s=1}^{r} [C^s F_s(u)]^{\frac{8d(I_s)}{(D-2)<U_s,U_s>}} \left\{ [F_0(u-u_0)]^{\frac{2}{1-d_0}} e^{2Q^0 u + 2P^0} \left[F_0^{-2}(u-u_0)du^2 + d\Omega_{d_0}^2 \right] \right.$$
$$\left. - \prod_{s=1}^{r} [C^s F_s(u)]^{\frac{-8}{<U_s,U_s>}} e^{2Q^1 u + 2P^1} dt^2 + \sum_{i=2}^{n} \prod_{s=1}^{r} [C^s F_s(u)]^{\frac{-8\delta_{iI_s}}{<U_s,U_s>}} e^{2Q^i u + 2P^i} ds_i^2 \right\}. \tag{3.28}$$

2. The dilatonic scalar fields are the following

$$\varphi^\alpha(u) = \sum_{s=1}^{r} \frac{4\chi_s \lambda_{a_s}^\alpha}{<U_s,U_s>} \ln[C^s F_s(u)] + uQ^{n+\alpha} + P^{n+\alpha}, \; \alpha = 1,\dots,\omega. \tag{3.29}$$

3. For scalar functions $\dot{\Phi}^s(u)$ we get

$$\dot{\Phi}^s(u) = Q_s e^{<U_s,x>} = Q_s \prod_{s'=1}^{m} \left[C^{s'} F_{s'}(u) \right]^{-C_{ss'}}, \; s = 1,\dots,r. \tag{3.30}$$

The corresponding F^a-field forms look as follows

$$F^{(a_s,e,I_s)} = \dot{\Phi}^{(a_s,e,I_s)} du \wedge \tau(I_s) \tag{3.31}$$

for the electrically charged p-brane and

$$F^{(a_s,m,I_s)} = Q_s \tau_0 \wedge \tau_{i_1} \wedge \dots \wedge \tau_{i_c}, \; \{i_1,\dots,i_c\} = \bar{I}_s, \; c = n_{a_s} - d_0 \tag{3.32}$$

for the p-brane with magnetic-type charge. We put $F^{(a_s,m,I_s)} = Q_s \tau_0$ if $\bar{I}_s = \emptyset$. We stress that if $r \equiv |S_*| = n + \omega$, then one must put $Q^A = P^A = 0$, $A = 0,\dots,n+\omega$ in this solution.

11

4 General solutions for models associated with Lie algebras

Now we list general solutions to the set of equations (3.23) for some special matrices $(C_{ss'})$.

1. $(C_{ss'}) = \text{diag}(2, \ldots, 2)$ is the Cartan matrix of the semi-simple Lie algebra $A_1 \oplus \ldots \oplus A_1$ of rank r. In this case the set of the characteristic vectors U_s is orthogonal.

$$F_s(u) \equiv e^{-q^s(u)} = \frac{\sin[w_s(u - u_{01})]}{w_s}, \quad s = 1, \ldots, r, \tag{4.1}$$

$$E_T = \sum_{s=1}^{r} \frac{2w_s^2}{< U_s, U_s >}, \tag{4.2}$$

   where w_s are arbitrary constants, which may be real (including zero) or imaginary.

2. $(C_{ss'}) = (2\delta_{ss'} - \delta_{s,s'+1} - \delta_{s,s'-1})$ is the Cartan matrix of the simple Lie algebra $A_r \equiv sl(r+1, C)$. In this case all characteristic vectors U_s are space-like with coinciding lengths, i.e.

$$< U_s, U_s > \equiv U^2, \quad s = 1, \ldots, r. \tag{4.3}$$

   By the transformation

$$q^s \mapsto q^s - \frac{\pi i}{2} m_s, \tag{4.4}$$

   where

$$m_s = 2 \sum_{s'=1}^{r} C^{ss'} = s(r + 1 - s), \tag{4.5}$$

   we put the set of equations (3.21) into the form

$$\ddot{q^s} = -\exp\left[\sum_{s'=1}^{r} C_{ss'} q^{s'}\right], \quad s, = 1, \ldots, r. \tag{4.6}$$

   These are precisely the A_r Toda equations [28]. Using the general solutions to these equations presented by Anderson [29], we obtain the following result

$$F_s(u) = i^{s(r+1-s)} \sum_{\mu_1 < \ldots < \mu_s}^{r+1} v_{\mu_1} \cdots v_{\mu_s} \Delta^2(\mu_1, \ldots, \mu_s) e^{(w_{\mu_1} + \ldots + w_{\mu_s})u}, \tag{4.7}$$

$$E_T = \frac{1}{2} \sum_{\mu=1}^{r+1} w_\mu^2. \tag{4.8}$$

   where $\Delta^2(\mu_1, \ldots, \mu_s)$ denotes the square of the Vandermonde determinant

$$\Delta^2(\mu_1, \ldots, \mu_s) = \prod_{\mu_i < \mu_j} \left(w_{\mu_i} - w_{\mu_j}\right)^2, \quad \Delta^2(\mu_1) \equiv 1. \tag{4.9}$$

   The constants v_μ and w_m, $\mu = 1, \ldots, r + 1$, have to satisfy the following constraints:

$$\prod_{\mu=1}^{r+1} v_\mu = \Delta^{-2}(1, \ldots, r + 1), \quad \sum_{\mu=1}^{r+1} w_\mu = 0. \tag{4.10}$$

   The constants v_μ, w_μ are in general complex. There are additional constraints on them if one requires the functions $F_s(u)$ and the integral of motion (4.8) to be real. In (4.7) we used $m_s = s(r + 1 - s)$ for A_r.

3. $(C_{ss'})$ is the following matrix

$$C_{ss'} = \begin{cases} 2\delta_{ss'} - \delta_{s,s'+1} - \delta_{s,s'-1} & \text{for} \quad s = 1,\dots,r, \ s' = 1,\dots,r-1, \\ \delta_{ss'} - \delta_{s,s'-1} & \text{for} \quad s = 1,\dots,r, \ s' = r. \end{cases} \qquad (4.11)$$

The Cartan matrix of the simple Lie algebra $B_r \equiv so(2r+1)$ may be obtained from that given in (4.11) by multiplying the last column of (C_{sr}) by 2. In this case the general solution to the set of equations (3.21) may be obtained from the previous formulae (4.7), (4.8) as in [30]. Notice that the equations (3.21) are symmetric under the following permutations $q^s \leftrightarrow q^{r+1-s}$ for $s = 1,\dots,r$ if $(C_{ss'})$ if the Cartan matrix of A_r. This implies that there are solutions (4.7) with $q^s \equiv q^{r+1-s}$ for $s = 1,\dots,r$. Moreover, this identification for $r = 4,6,8,\dots$ leads to the $(r/2)$ equations of the form (3.21) with the matrix (4.11). Consequently the general solution of the equations (3.21) with the matrix (4.11) for $r = r_0$ may be obtained form (4.7) for $r = 2r_0$ by putting additional constraints on the constants v_μ, w_μ providing with the identities $F_s(u) \equiv F_{2r_0+1-s}(u), \ s = 1,\dots,2r_0$.

4. $(C_{ss'})$ is the Cartan matrix of the simple Lie algebra $C_r \equiv sp(r,C)$, i.e.

$$C_{ss'} = \begin{cases} 2\delta_{ss'} - \delta_{s-1,s'} - \delta_{s-1,s'} & \text{for} \quad s = 1,\dots,r-1, \ s' = 1,\dots,r, \\ \delta_{ss'} - \delta_{s,s'-1} & \text{for} \quad s = r, \ s' = 1,\dots,r. \end{cases} \qquad (4.12)$$

In this case the general solution to the set of equations (3.21) with $r = r_0$ may be obtained from (4.7) with $r = 2r_0 - 1$ by putting additional constraints on the constants v_μ, w_μ providing with identities $F_s(u) \equiv F_{2r_0-s}(u), \ s = 1,\dots,2r_0 - 1$. It stems ¿from the following property of the set (3.21): the identification $q^s \equiv q^{2r_0-s}$ for $s = 1,\dots,2r_0 - 1$ reduces the set (3.21) with the Cartan matrix of A_{2r_0-1} to the set (3.21) with the Cartan matrix of C_{r_0} $(r_0 \geq 2)$.

5 The particular solution describing black holes

Suppose the nondegenerate Cartan-type matrix $(C_{ss'})$ satisfies the conditions

$$m_s = 2\sum_{s'=1}^{r} C^{ss'} > 0, \ s = 1,\dots,r. \qquad (5.1)$$

The conditions are valid for extremely large class of the p-brane models. For instance, the parameters m_s are natural numbers if $(C_{ss'})$ is the Cartan matrix of a semi-simple Lie algebra G [25]. For $G = A_r = sl(r+1,C)$ the parameters m_s are given by (4.5).

Under the conditions (5.1) the set of equations (3.23) admits the following particular solution

$$F_s(u) = a_s \left(\frac{\sinh[\bar{\mu}(u - u_{01})]}{\bar{\mu}} \right)^{m_s}, \ s = 1,\dots,r, \qquad (5.2)$$

where the constants a_s are defined by

$$a_s = \prod_{s'=1}^{r} (m_{s'})^{-C^{ss'}} \qquad (5.3)$$

and the constants $\bar{\mu}, u_{01}$ are arbitrary. The corresponding to (5.2) integral of motion (3.10) has the form

$$E_T = \sum_{s,s'=1}^{r} \frac{C_{ss'}}{<U_s,U_s>} \frac{\dot{F}_s}{F_s} \frac{\dot{F}_{s'}}{F_{s'}} - \sum_{s=1}^{r} \frac{2}{<U_s,U_s>} \prod_{s'=1}^{r} (F_{s'})^{-C_{ss'}} = 2\bar{\mu}^2 \sum_{s=1}^{r} \frac{m_s}{<U_s,U_s>}. \qquad (5.4)$$

13

For $\bar{\mu} = 0$ the formulas (5.2),(5.4) read

$$
\begin{aligned}
F_s(u) &= a_s(u - u_{01})^{m_s}, \quad s = 1, \ldots, r, & (5.5) \\
E_T &= 0. & (5.6)
\end{aligned}
$$

It is evident that (5.5) represents the polynomials in the radial coordinate u if the parameters m_s are natural numbers. As we have already mentioned, m_s are natural numbers if, for instance, $(C_{ss'})$ is the Cartan matrix of a semi-simple Lie algebra. The formula (5.5) does not exhaust all possible polynomial solutions to the set (3.23). As far as we know, an explicit general form of the polynomial solution, which appears for the set of equations (3.23) with arbitrary natural numbers m_s and vanishing integral of motion (5.4), is not found. There are only few examples in the literature. For instance, in [4] all possible polynomial solutions were obtained for the matrix $(C_{ss'})$ supposed to be the Cartan matrix of the Lie algebras $A_r \equiv sl(r+1, C)$ with $r = 1, 2, 3$. It is easy to check that an arbitrary polynomial solution to (3.23) under the condition $E_T = 0$ may be obtained from (5.5) by adding some polynomial of lower degree to the leading term $a_s u^{m_s}$.

Now we use the particular solution (5.2) with $\bar{\mu} > 0$ and its special form (5.5) corresponding to $\bar{\mu} = 0$ for constructing non-extremal and extremal black holes, respectively. Consider the general form of exact solution (3.28)-(3.31) under the following additional assumptions

1. We put

$$
Q^A = \bar{\mu} \left(\frac{2V_0^A}{< V_0, V_0 >} + \sum_{s=1}^{r} \frac{2m_s U_s^A}{< U_s, U_s >} - \delta_1^A \right), \quad A = 0, \ldots, n + \omega. \quad (5.7)
$$

One may verify that the conditions $< Q, V_0 >= 0$, $< Q, U_s >= 0$, $s = 1, \ldots, r$ are valid. Using the zero-energy constraint (3.6) and (5.4), we find the constant E_0

$$
E_0 = -E_T - \frac{1}{2} < Q, Q >= -\frac{1}{2} d_0 \bar{d}_0 \mu^2, \quad (5.8)
$$

where we denoted

$$
\mu = \bar{\mu}/\bar{d}_0, \quad \bar{d}_0 = d_0 - 1. \quad (5.9)
$$

2. We take the parameters P^A in the form

$$
\left(P^A \right) = \sum_{s=1}^{r} \ln[C^s F_s(u_0)] \frac{2 \left(U_s^A \right)}{< U_s, U_s >} - u_0 \left(Q^A \right) + \ln \varepsilon_0 (1, 0, \underbrace{-R^2, \ldots, R^n}_{n-1}, \underbrace{0, \ldots, 0}_{\omega}), \quad (5.10)
$$

where ε_0 is an arbitrary positive constant. The conditions $< P, V_0 >= 0$ lead to the following constraint on parameters $R^2, \ldots, R^n$

$$
\sum_{i=2}^{n} R^i d_i = \bar{d}_0. \quad (5.11)
$$

Combining the conditions $< P, U_s >= 0, s = 1, \ldots, r$ and (3.13), we get

$$
Q_s^2 = \frac{< U_s, U_s >}{4} \varepsilon_0^{-2 \sum_{i=2}^{n} d_i \delta_{i I_s} R^i} \prod_{s'=1}^{r} [F_{s'}(u_0)]^{-C_{ss'}}, \quad s = 1, \ldots, r. \quad (5.12)
$$

14

3. Now we consider the solution for $u \in (u_0, +\infty)$ and introduce the following new radial coordinate

$$R = \frac{R_0}{1 - \exp[-2\bar{\mu}(u - u_0)]} = \varepsilon_0 \left(\frac{2\mu}{1 - \exp[-2\bar{\mu}(u - u_0)]} \right)^{1/\bar{d}_0} , \quad R > R_0. \qquad (5.13)$$

The constant R_0 is defined by

$$R_0 = \varepsilon_0 (2\mu)^{1/\bar{d}_0}. \qquad (5.14)$$

Here we take the constant u_{01} in (5.2) such that $(u_0 - u_{01}) > 0$. Moreover we introduce the constant

$$R_* = R \mid_{u=2u_0-u_{01}} = \varepsilon_0 \left(\frac{2\mu}{1 - \exp[-2\bar{\mu}(u_0 - u_{01})]} \right)^{1/\bar{d}_0} > R_0. \qquad (5.15)$$

Finally, we obtain the metric

$$
\begin{aligned}
ds^2 &= \left[1 + \left(\frac{R_*}{R} \right)^{\bar{d}_0} - \left(\frac{R_0}{R} \right)^{\bar{d}_0} \right]^{\sum_{s=1}^{r} \frac{8 m_s d(I_s)}{(D-2) <U_s, U_s>}} \left\{ \frac{dR^2}{1 - (R_0/R)^{\bar{d}_0}} + R^2 d\Omega_{\bar{d}_0}^2 \right. \\
&\quad - \left[1 + \left(\frac{R_*}{R} \right)^{\bar{d}_0} - \left(\frac{R_0}{R} \right)^{\bar{d}_0} \right]^{-\sum_{s=1}^{r} \frac{8 m_s}{<U_s, U_s>}} \left(1 - \left(\frac{R_0}{R} \right)^{\bar{d}_0} \right) dt^2 \\
&\quad + \left. \sum_{i=2}^{n} \varepsilon_0^{-2R^i} \left[1 + \left(\frac{R_*}{R} \right)^{\bar{d}_0} - \left(\frac{R_0}{R} \right)^{\bar{d}_0} \right]^{-\sum_{s=1}^{r} \frac{8 m_s \delta_{iI_s}}{<U_s, U_s>}} ds_i^2 \right\},
\end{aligned}
\qquad (5.16)
$$

the dilatonic scalar fields

$$\varphi^\alpha = \sum_{s=1}^{r} \frac{4 m_s \chi_s \lambda_{a_s}^\alpha}{<U_s, U_s>} \ln \left[1 + \left(\frac{R_*}{R} \right)^{\bar{d}_0} - \left(\frac{R_0}{R} \right)^{\bar{d}_0} \right], \quad \alpha = 1, \ldots, \omega, \qquad (5.17)$$

and the potential derivatives

$$
\begin{aligned}
\frac{d\Phi^s}{dR} &= - \operatorname{sgn}(Q_s) 2\bar{d}_0 \sqrt{ \frac{m_s}{<U_s, U_s>} \left(1 - \left(\frac{R_0}{R_*} \right)^{\bar{d}_0} \right) } \\
&\quad \times \varepsilon_0^{-\sum_{i=2}^{n} d_i \delta_{iI_s} R^i} \left(\frac{R_*}{R} \right)^{\bar{d}_0} \left[1 + \left(\frac{R_*}{R} \right)^{\bar{d}_0} - \left(\frac{R_0}{R} \right)^{\bar{d}_0} \right]^{-2}.
\end{aligned}
\qquad (5.18)
$$

The corresponding F^a-field forms may be obtained by (3.31),(3.32), where

$$|Q_s| = 2\bar{d}_0 \sqrt{ \frac{m_s}{<U_s, U_s>} \left(1 - \left(\frac{R_0}{R_*} \right)^{\bar{d}_0} \right) } \varepsilon_0^{\sum_{i=2}^{n} d_i \delta_{iI_s} R^i} \left(\frac{R_*}{\varepsilon_0} \right)^{\bar{d}_0}. \qquad (5.19)$$

Then constants $\bar{\mu}, \varepsilon_0, (u_0 - u_{01})$ are independent. The constants $\mu, \bar{d}_0, R_0, R_*$ are defined by (5.9),(5.14),
(5.15). The parameters $R^2, \ldots, R^n$ obey the relation (5.11).

Now we analyze the particular solution (5.16)-(5.18). The metric (5.16) is asymptotically flat, i.e.

$$\lim_{R \to +\infty} ds^2 = dR^2 + R^2 d\Omega_{\bar{d}_0}^2 - dt^2 + \sum_{i=2}^{n} \varepsilon_0^{-2R^i} ds_i^2. \qquad (5.20)$$

15

According to (5.11) all parameters $R^2, \ldots, R^n$ may be positive. Then, the constant scale factors $\varepsilon_0^{-2R^i}$ of internal spaces $M_2, \ldots, M_n$ are arbitrary small if ε_0 is large enough.

If $\bar{\mu} > 0$ the particular solution describes a non-extremal black hole with the horizon at $R = R_0$. The active gravitational mass M_g and the Hawking temperature T_H of this black hole read

$$2G_N M_g = R_*^{\bar{d}_0} \left[\left(1 - \left(\frac{R_0}{R_*} \right)^{\bar{d}_0} \right) \sum_{s=1}^{r} \frac{4m_s U_s^1}{< U_s, U_s >} + \left(\frac{R_0}{R_*} \right)^{\bar{d}_0} \right], \qquad (5.21)$$

$$T_H = \frac{\bar{d}_0}{4\pi k_B R_*} \left(\frac{R_0}{R_*} \right)^{\sum_{s=1}^{r} \frac{4m_s}{<U_s,U_s>} - 1}, \qquad (5.22)$$

where G_N and k_b are Newton's gravitational constant and Boltzmann's constant, respectively.

The solution (5.16)-(5.18) may be considered in the so-called extreme case, when $\bar{\mu} = 0$ ($R_0 = 0$). It follows from general statements proved in [17] that the point $R = 0$ is a curvature singularity of the metric (5.16) with $R_0 = 0$ if $T_H \to +\infty$ as $R_0 \to +0$. Then, the particular solution admits an extremal black hole only if

$$\sum_{s=1}^{r} \frac{4m_s}{< U_s, U_s >} \geq 1. \qquad (5.23)$$

References

[1] *K.S. Stelle.* "Lectures on Supergravity p-branes ", hep-th/9701088.

[2] *M.J. Duff, H. Lü, and C.N. Pope.* "The Black Branes of M-theory", hep-th/9604052.

[3] *H. Lü, C.N. Pope, K.W. Xu.* "Liouville and Toda Solitons in M-theory", hep-th/9604058.

[4] *H. Lü, and C.N. Pope.* "$SL(N + 1, R)$ Toda Solitons in Supergravities", hep-th/9607027.

[5] *H. Lü, and C.N. Pope.* "Black p-branes and Their Vertical Dimensional Reduction", hep-th/9609126.

[6] *M. Cvetic, A.A. Tseytlin.* Nucl. Phys. B. 1996. V. 478. P. 181.

[7] *V.D. Ivashchuk, V.N. Melnikov.* "Intersecting p-brane Solutions in Multidimensional Gravity and M-theory", hep-th/9612089; Gravitation & Cosmology. 1996. V. 2. P. 297.

[8] *I.Ya. Aref'eva, O.A. Rytchkov.* "Incidence Matrix Description of Intersecting p-brane Solutions", hep-th/9612236.

[9] *I.Ya. Aref'eva, M.G. Ivanov, O.A. Rytchkov.* "Properties of Intersecting p-branes in Various Dimensions", hep-th/9702077.

[10] *I.Ya. Aref'eva, M.G. Ivanov, I.V. Volovich.* "Non-extremal Intersecting p-branes in Various Dimensions", hep-th/9702079.

[11] *N. Ohta and T. Shimuzu.* " Non-extreme Black Holes from Intersecting M-branes", hep-th/9701095.

[12] *N. Ohta.* "Intersection Rules for Non-extreme p-branes", hep-th/9702164.

[13] *V.D. Ivashchuk, V.N. Melnikov.* Phys. Lett. B. 1997. V. 403. P. 23.

[14] *J.M. Maldacena.* "Black Holes and D-branes", hep-th/9705078.

[15] *V.D. Ivashchuk, V.N. Melnikov.* "Multidimensional Classical and Quantum Cosmology with Intersecting p-branes", hep-th/9708157; J. Math. Phys. 1998. V. 39. P. 2866.

[16] *V.D. Ivashchuk, V.N. Melnikov.* "Sigma-model for the Generalized Composite p-branes", hep-th/9705036; Class. Quantum Grav. 1997. V. 14. P. 3001.

[17] *K.A. Bronnikov, V.D. Ivashchuk, V.N. Melnikov.* "The Reissner-Nordström Problem for Intersecting Electric and Magnetic p-branes", gr-qc/9710054; Gravitation & Cosmology. 1997. V. 3. P. 203.

[18] M.A. Grebeniuk and V.D. Ivashchuk, "Sigma-model Solutions and Intersecting p-branes Related to Lie Algebras", hep-th/9805113; Phys. Lett. B 1998. V. 442.P.125.

[19] *K.A. Bronnikov, U. Kasper, M. Rainer.* "Intersecting Electric and Magnetic p-branes: Spherically Symmetric Solutions", gr-qc/9708058; Gen. Relativity and Gravitation. 1999. V.31. P.1681.

[20] *K.A. Bronnikov,* "Block-orthogonal Brane Systems, Black Holes and Wormholes", hep-th/9710207; Gravitation & Cosmology. 1998. V. 4. P. 49.

[21] *V.D. Ivashchuk, V.N. Melnikov.* "Madjumdar-Papaptrou Type Solutions in Sigma Model and Intersecting p-branes", hep-th/9802121; Class. Quantum Grav. 1999. V. 16. P. 849.

[22] *V.D. Ivashchuk,S.-W. Kim, V.N. Melnikov.* "Hyperbolic Kac-Moody Algebra from Intersecting p-branes", hep-th/9803006; J. Math Phys. 1999.V.40. P. 4072.

[23] *V.D. Ivashchuk and V.N. Melnikov.* "Multidimensional Cosmological and Spherically Symmetrical Solutions with Intersecting p-branes", gr-qc/9901001; In: Lecture Notes in Physics, V. 537. Mathematical and Quantum Aspects Of Relativity and Cosmology. Eds. S. Cotsakis and G. Gibbons. Proc. 2nd Samos Meeting (September 1998, Pithagoreon), Springer, Berlin, 2000. P. 214.

[24] *S. Cotsakis, V.D. Ivashchuk and V.N. Melnikov.* "p-brane Black Holes and Post-Newtonian Approximation", hep-th/9902148; Gravitation & Cosmology. 1999. V. 5. P. 52.

[25] *V.D. Ivashchuk and V.N. Melnikov.* p-brane Black Hole Solution for General Intersection Rules", hep-th/9910041; Gravitation & Cosmology. 1999. V.5. P. 313.

[26] *E. Cremmer, B. Julia, J. Scherk.* Phys. Lett. B. 1978. V. 76. P. 409.

[27] *V.R. Gavrilov, V.D. Ivashchuk, V.N. Melnikov* J. Math. Phys. 1995. V. 36. P. 5829.

[28] *M. Toda.* Theory of Nonlinear Lattices. Springer-Verlag, Berlin. 1981

[29] *A. Anderson.* J. Math. Phys. 1996. V. 37. P. 1349.

[30] *A.N. Leznov and M.V. Saveliev.* Group-theoretical Methods for Integration of Nonlinear Dynamical Systems. Birkhauser, Basel, 1992.